$$D_5=C^5-4C^3+3C,$$
$$D_6=C^6-5C^4+6C^2-1,$$
$$D_7=C^7-6C^5+10C^3-4C,$$
$$D_8=C^8-7C^6+15C^4-10C^2+1.$$

It is clear that starting with D_n and reading the terms diagonally we have the expansion of $(C-1)^n$. For instance, starting with D_4 we have $C^4-4C^3+6C^2-4C+1$. Hence, reading horizontally, the 1, 2, 3, 4, etc., terms of D_n will be the 1, 2, 3, 4, etc., terms in the expansions of $(C-1)^n$, $(C-1)^{n-1}$, $(C-1)^{n-2}$, $(C-1)^{n-3}$, etc., respectively. The rth term will be the rth term of the expansion of $(C-1)^{n-r+1}$. Hence

$$D_n=C^n-(n-1)C^{n-2}+\frac{(n-2)(n-3)}{2!}C^{n-4}-\frac{(n-3)(n-4)(n-5)}{3!}C^{a-6}+\ldots$$

$$\text{or } D_n=C^n+\sum_{r=1}^{n}(-1)^r\frac{(n-r)(n-r-1)\ldots(n-2r+1)}{r!}C^{n-2r}.$$

A very neat solution of this problem was also received from a contributor who failed to sign his name to the solution. Will contributors please note that we wish them to put their names to the solutions and to observe the order of the printed solutions, viz., put name at beginning of solution rather than at the end?

GEOMETRY.

369. Proposed by W. J. GREENSTREET, A. M., Editor, Mathematical Gazette, Stroud, England.

Prove by inversion that if two circles cut at a given angle, touch each a given circle, and pass each through the same fixed point, then shall the envelope of the points of contact be a conic.

Discussion by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Since all of the contact points lie on the fixed circles it seems probable that the desired locus is that of the second point of intersection of the variable circles. In Fig. 1, let S_1 and S_2 be the fixed circles, T_1 and T_2 the variable circles passing through the fixed point Q at the constant angle θ.

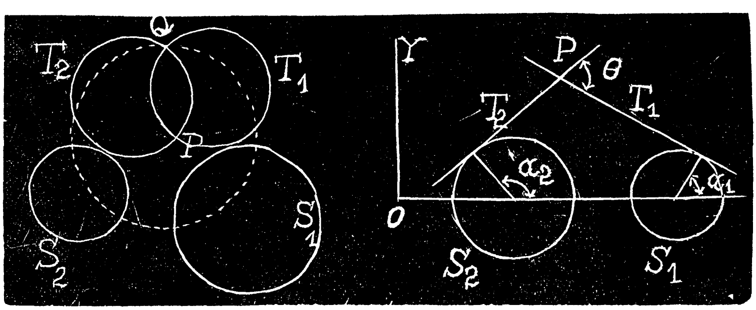

Fig. 1. Fig. 2.

Then the locus to be found is that of the second intersection P. Through the fixed circles and Q considered as a point circle, Fig. 1, there may be passed one orthogonal circle whose center is their radical center. With Q as the center of inversion and with any convenient radius, Fig. 1 is to be inverted, thus changing T_1 and T_2 into straight lines in Fig. 2, crossing at the constant angle θ, and tangent to the new fixed circles S_1 and S_2.

The orthogonal circle mentioned above inverts into the straight line through the centers of S_1 and S_2.

The analytic work from this stage is as follows: The equations of S_1 and S_2 may be taken as

$$(x-a_1)^2+y^2=r_1^2 \text{ and } (x-a_2)^2+y^2=r_2^2,$$

hence the equations of T_1 and T_2 are

$$(x-a_1)\cos\alpha_1+y\sin\alpha_1=r_1 \text{ and } (x-a_2)\cos\alpha_2+y\sin\alpha_2=r_2,$$

in which $\alpha_1-\alpha_2=\theta=$constant, this being the angle condition. Replacing θ by 2ϕ and writing $\alpha_1=\alpha+\phi$, $\alpha_2=\alpha-\phi$, the equations of T_1 and T_2 become

$$(x-a_1)\cos(\alpha+\phi)+y\sin(\alpha+\phi)=r_1, \quad (x-a_2)\cos(\alpha-\phi)+y\sin(\alpha-\phi)=r_2.$$

Expanding the trigonometric functions and solving the two equations for $\cos\alpha$ and $\sin\alpha$ leads to the elimination of α, and gives the following for the locus of P:

$$\begin{aligned}&\{[(x-a_1)(x-a_2)+y^2]\sin\theta-y(a_1-a_2)\cos\theta\}^2\\&=r_1^2[(x-a_2)^2+y^2]+r_2^2[(x-a_1)^2+y^2]\\&-2r_1r_2\{[(x-a_1)(x-a_2)+y^2]\cos\theta+y(a_1-a_2)\sin\theta\}.\end{aligned}$$

Two special cases of the preceding equation may best be considered at this time. If S_1 and S_2 in Fig. 2 are coincident or even merely concentric, then the locus of P is a circle by elementary geometry, hence the locus of P in Fig. 1 is also a circle. This case occurs when Q, the fixed point in Fig. 1, is either of the point circles of the family of circles having a common radical axis determined by S_1 and S_2. From the general equation this result appears by placing $a_1=a_2$, and then the locus degenerates into a point circle and the circle mentioned above. Again the general equation degenerates when $r_1=r_2=0$, giving

$$\{[(x-a_1)(x-a_2)+y^2]\sin\theta-y(a_1-a_2)\cos\theta\}^2=0,$$

which is the circle, doubly counted, resulting from making S_1 and S_2 poin circles in both figures. As in the preceding case, the locus of P in Fig. 1 is also a circle. The general locus is a bicircular quartic, hence its inverse is also a bicircular quartic, thus determining the locus of P (Fig. 1) which constitutes the solution of the given problem.

NOTE. No solutions of 374 and 376 have yet been received. We shall be pleased to have our contributors take up these two problems for solution. ED. F.

CALCULUS.

300. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Solve the differential equation obtaining the complete primitive,

$$(x^2+x^2y+2xy-y^2-y^3)dx+(y^2+xy^2+2xy-x^2-x^3)dy=0.$$

Solution by V. M. SPUNAR, Chicago, Ill., and the PROPOSER.

If F is an integrating factor of the differential equation

$$Mdx+Ndy=0,$$

we have the relation

$$\frac{F'(v)}{F(v)}=\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N\dfrac{\partial v}{\partial x}-M\dfrac{\partial v}{\partial y}}=\frac{4(x-y)(x+y+1)}{N\dfrac{\partial v}{\partial x}-M\dfrac{\partial v}{\partial y}}.$$

On account of the symmetry of M and N, it is evident that v is symmetrical in x and y. Trying $v=x+y$, we have

$$\frac{F'(v)}{F(v)}=-\frac{4(x+y+1)}{(x+y)^2+2(x+y)}=-\frac{4(v+1)}{v^2+2v}.$$

Integrating, we have,

$$F=\frac{1}{v^2(v+2)^2}=\frac{1}{(x+y)^2(x+y+2)^2}.$$

Trying, also, $v=(1+x)(1+y)$, we have,

$$\frac{F'(v)}{F(v)}=-\frac{2}{x+xy+y}=-\frac{2}{v-1}.$$

Integrating,

$$F=\frac{1}{(v-1)^2}=\frac{1}{(x+xy+y)^2}.$$